

The Quantum Bit Commitment Theorem

Jeffrey Bub*

Philosophy Department, University of Maryland, College Park, MD 20742, USA

Abstract

Unconditionally secure two-party bit commitment based *solely* on the principles of quantum mechanics (without exploiting special relativistic signalling constraints, or principles of general relativity or thermodynamics) has been shown to be impossible, but the claim is repeatedly challenged. The quantum bit commitment theorem is reviewed here and the central conceptual point, that an ‘Einstein-Podolsky-Rosen’ attack or cheating strategy can always be applied, is clarified. The question of whether following such a cheating strategy can ever be disadvantageous to the cheater is considered and answered in the negative. There is, indeed, no loophole in the theorem.

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1 Introduction

Over the past few years, the new fields of quantum information, quantum computation, and quantum cryptology have emerged as the locus of foundational research in quantum mechanics. In quantum cryptology, the main results have been a variety of provably secure protocols for key distribution, following the original Bennett and Brassard (BB84) protocol [5], and an important ‘no go’ theorem by Mayers [26, 27, 20]: the impossibility of

*Email address: jbub@carnap.umd.edu

unconditionally secure two-party bit commitment based *solely* on the principles of quantum mechanics (without exploiting special relativistic signalling constraints, or principles of general relativity or thermodynamics). The quantum bit commitment theorem generalizes previous results restricted to one-way communication protocols by Mayers [25] and by Lo and Chau [19], and applies to quantum, classical, and quantum-classical hybrid schemes (since classical information is essentially quantum information subject to certain constraints). The restriction to two-party schemes excludes schemes that involve a trusted third-party or trusted channel properties, and the restriction to schemes based solely on the principles of quantum mechanics excludes schemes that exploit special relativistic signalling constraints (see below), or schemes that might involve time machines or black holes.

In a key distribution protocol, the object is for two parties, Alice and Bob, who initially share no information, to exchange information via quantum and classical channels, so as to end up sharing a secret key (which they can then use for encryption), in such a way as to ensure that any attempt by an eavesdropper, Eve, to gain information about the secret key will be detected with non-zero probability.

The features of quantum mechanics that allow secure key distribution are, essentially, the quantum ‘no cloning’ theorem (which makes it impossible for Eve to copy quantum communications between Alice and Bob for later analysis), and the fact that nonorthogonal quantum states cannot be distinguished without disturbing the states, so any information gain that depends on distinguishing such states must introduce some detectable disturbance.

In a bit commitment protocol, one party, Alice, supplies an encoded bit to a second party, Bob. The information available in the encoding should be insufficient for Bob to ascertain the value of the bit, but sufficient, together with further information supplied by Alice at a subsequent stage when she is supposed to reveal the value of the bit, for Bob to be convinced that the protocol does not allow Alice to cheat by encoding the bit in a way that leaves her free to reveal either 0 or 1 at will.

To illustrate the idea, suppose Alice claims the ability to predict advances or declines in the stock market on a daily basis. To substantiate her claim without revealing valuable information (perhaps to a potential employer, Bob) she suggests the following demonstration: She proposes to record her prediction, before the market opens, by writing a 0 (for ‘decline’) or a 1 (for ‘advance’) on a piece of paper, which she will lock in a safe. The safe will be handed to Bob, but Alice will keep the key. At the end of the day’s

trading, she will announce the bit she chose and prove that she in fact made the commitment at the earlier time by handing Bob the key. The question is whether there exists a quantum analogue of this procedure that is unconditionally secure: provably secure by the laws of physics against cheating by either Alice or Bob. Note that Bob can cheat if he can obtain *some* information about Alice's commitment before she reveals it (which would give him an advantage in repetitions of the protocol with Alice). Alice can cheat if she can delay actually making a commitment until the final stage when she is required to reveal her commitment, or if she can change her commitment at the final stage with a very low probability of detection.

The importance of quantum bit commitment as a cryptological primitive arises because of its relation to other cryptological protocols. Lo [22] has argued that the impossibility of unconditionally secure quantum bit commitment implies the impossibility of secure quantum one-sided two-party computations, and hence the impossibility of secure quantum one-out-of-two oblivious transfer. It is easy to see that a remote coin tossing procedure, in which neither party can cheat, would be possible if secure bit commitment were possible, which would allow unconditionally secure remote gambling (gambling over the internet, for example). But note that a procedure for remote fair games has been proposed by Goldenberg, Vaidman, and Wiesner [13], so this is a weaker protocol than bit commitment.

Bennett and Brassard originally proposed a quantum bit commitment protocol in [5]. The basic idea was to associate the 0 and 1 commitments with two statistically equivalent quantum mechanical mixtures (represented by the same density operator). As they showed in the same paper, Alice can cheat by adopting an 'Einstein-Podolsky-Rosen' (EPR) attack or cheating strategy: she prepares entangled pairs of particles, keeps one of each pair (the ancilla) and sends the second particle (the channel particle) to Bob. In this way she can fake sending one of two equivalent mixtures to Bob and reveal either bit at will at the opening stage by effectively creating the desired mixture via appropriate measurements on her ancillas. Bob cannot detect this cheating strategy.

In a later paper [7], Brassard, Crépeau, Josza, and Langlois proposed a bit commitment protocol that they claimed to be unconditionally secure. The BCJL scheme was first shown to be insecure by Mayers [24, 25]. Subsequently, Mayers [26, 27] and Lo and Chau [19] independently showed that a large class of quantum bit commitment schemes are insecure. Lo and Chau presented their result in [19] as applicable only to all proposed quantum bit

commitment schemes, including the BCJL scheme (for which they relied on Mayers' extended analysis in [25]). But as Mayers showed in [26, 27], the insight of Bennett and Brassard in [5] can be extended to a proof that a generalized version of the EPR cheating strategy can always be applied, if the Hilbert space is enlarged in a suitable way by introducing additional ancilla particles. Following Mayers, a similar result is proved in Lo and Chau [20], where the operative assumption is that both Alice and Bob have available quantum computers of unlimited power and are capable of storing quantum signals indefinitely.

Mayers' analysis in [27] explicitly models the exchange of quantum and classical information in two-way quantum bit commitment protocols via a 'direct' approach. For an interesting 'indirect' or 'reduction' approach, see [8, 9, 10]. Classical information can be understood as a type of quantum information with additional constraints. The distinction between classical and quantum information was always explicit in the analysis of proposed quantum bit commitment protocols. According to Mayers (personal communication), this explains why researchers failed to see the general impossibility of quantum bit commitment, even after the basic mathematical result, which is valid in a purely quantum world, was known.

The negative results of Mayers and Lo and Chau came as a surprise and were received with dismay by the quantum cryptology community. The proof of the basic theorem, which exploits the biorthogonal decomposition theorem, is remarkably simple, but the impossibility of secure bit commitment based solely on the principles of quantum (or classical) mechanics has profound consequences. Indeed, it would not be an exaggeration to say that the significance of the quantum bit commitment theorem is comparable to Bell's locality theorem [4] for quantum mechanics. Brassard and Fuchs have speculated (private communication and [12]) that quantum mechanics can be derived from two postulates about quantum information: the possibility of secure key distribution and the impossibility of secure bit commitment. That is, in a quantum world the communication of information is characterized precisely in this way in terms of a limited sort of privacy.

Perhaps because of the simplicity of the proof and the universality of the claim, the quantum bit commitment theorem is continually challenged in the literature (see, for example, [11, 28, 31]), on the basis that the proof does not cover all possible procedures that might be exploited to implement quantum bit commitment. There seems to be a general feeling that the theorem is 'too good to be true' and that there must be a loophole.

In fact, there is no loophole. While Kent [16, 17] has shown how to implement a secure classical bit commitment protocol by exploiting relativistic signalling constraints in a timed sequence of communications between verifiably separated sites for both Alice and Bob, and Hardy and Kent [14] and Aharonov, Ta-Shma, Vazirani, and Yao [2] have investigated the security of ‘cheat-sensitive’ or ‘weak’ versions of quantum bit commitment, these results are not in conflict with the quantum bit commitment theorem. In a bit commitment protocol as usually construed, there is a time interval of arbitrary length, where no information is exchanged, between the end of the commitment stage of the protocol and the opening or unveiling stage, when Alice reveals the value of the bit. Kent’s ingenious scheme effectively involves a third stage between the commitment stage and the unveiling stage, in which information is exchanged between Bob’s sites and Alice’s sites at regular intervals until one of Alice’s sites chooses to unveil the originally committed bit. At this moment of unveiling the protocol is not yet complete, because a further sequence of unveilings is required between Alice’s sites and corresponding sites of Bob before Bob has all the information required to verify the commitment at a single site. If a bit commitment protocol is understood to require an arbitrary amount of ‘free’ time between the end of the commitment stage and the opening stage (in which no step is to be executed in the protocol), then the quantum bit commitment theorem covers protocols that exploit special relativistic signalling constraints. (I am indebted to Dominic Mayers for clarifying this point.) The aim of the following discussion will be to clarify the underlying logic of the proof, and especially the crucial significance of the assumption that both parties can be assumed to have access to quantum computers, so that a (generalized) EPR cheating strategy is always possible.

In Section 2, I review the structure of the proof and show how any step in a bit commitment protocol that requires Alice or Bob to make a determinate choice (whether to perform one of a number of alternative measurements, or whether to implement one of a number of alternative unitary transformations) can always be replaced by an EPR cheating strategy in the generalized sense, assuming that Alice and Bob are both equipped with quantum computers. That is, a classical disjunction over determinate possibilities—this operation *or* that operation—can always be replaced by a quantum entanglement and a subsequent measurement (perhaps at a more convenient time for the cheater) in which one of the possibilities becomes determinate. Essentially, the classical disjunction is replaced by a quantum disjunction. This

cheating strategy cannot be detected. Similarly, a measurement can be ‘held at the quantum level’ without detection: instead of performing the measurement and obtaining a determinate outcome as one of a number of possible outcomes, a suitable unitary transformation can be performed on an enlarged Hilbert space, in which the system is entangled with a ‘pointer’ ancilla in an appropriate way, and the procedure of obtaining a determinate outcome (which involves decoherence, or the ‘collapse’ of the quantum state onto an eigenstate of the observable measured) can be delayed. The possibility of keeping the series of transactions between Alice and Bob at the quantum level by enlarging the Hilbert space, until the final exchange of classical information when Alice reveals her commitment, is the crucial insight that underlies Mayers’ general proof. In John Smolin’s whimsical terminology, this is the doctrine of the Church of the Larger Hilbert Space: the belief that a fully quantum treatment can always be obtained by extending the Hilbert space.

If it can be assumed that a measurement has in fact been performed and a determinate outcome obtained, then secure bit commitment is possible. This is tantamount to assuming that an EPR cheating strategy is blocked. Since there is no way to distinguish whether the protocol has been followed or replaced by an EPR cheating strategy, it would seem that there is no way to ensure that a measurement has in fact been performed and a determinate outcome recorded.

But how do we know that there is no bit commitment protocol of the following sort: Suppose, at some stage of the protocol, Bob (say) is required to perform one of two alternative measurements, X or Y, chosen at random. If Bob actually chooses one of X or Y, and actually performs the measurement and obtains a determinate outcome, then the protocol is secure against cheating by both parties. If Bob implements an EPR strategy and keeps the choice and the measurement at the quantum level, then it turns out that Alice has a greater probability of cheating successfully than Bob. If there were such a protocol, then even though Bob could implement an EPR strategy without detection, he would effectively be forced to make the choice and carry out the measurement, since he would not choose to put himself in a weaker position relative to Alice over the long run in a series of bit commitment transactions. In Section 3, I show that the possibility of such a protocol is blocked by the theorem itself. That is, adopting an EPR cheating strategy is never disadvantageous to the cheater.

2 The Bit Commitment Theorem

Any bit commitment scheme will involve a series of transactions between Alice and Bob, where a certain number, n , of quantum systems—the ‘channel particles’—are passed between them and subjected to various operations (unitary transformations, measurements), possibly chosen randomly. I show now how these operations can always be replaced, without detection, by entangling a channel particle with one or more ancilla particles that function as ‘pointer’ particles for measurements or ‘dice’ particles for random choices. This is the (generalized) EPR cheating strategy.

Suppose, at a certain stage of a bit commitment protocol, that Bob is required to make a random choice between measuring one of two observables, X or Y , on each channel particle he receives from Alice. For simplicity, assume that X and Y each have two eigenvalues, x_1, x_2 and y_1, y_2 . After recording the outcome of the measurement, Bob is required to return the channel particle to Alice. When Alice receives the i ’th channel particle she sends Bob the next channel particle in the sequence. We may suppose that the measurement outcomes that Bob records form part of the information that enables him to confirm Alice’s commitment, once she discloses it (together with further information), so he is not required to report his measurement outcomes to Alice until the final stage of the protocol when she reveals her commitment.

Instead of following the protocol, Bob can construct a device that entangles the input state $|\psi\rangle_C$ of a channel particle with the initial states, $|d_0\rangle_B$ and $|p_0\rangle_B$, of two ancilla particles that he introduces, the first of which functions as a ‘quantum die’ for the random choice and the second as a ‘quantum pointer’ for the measurement. It is assumed that Bob’s ability to construct such a device—a special purpose quantum computer—is restricted only by the laws of quantum mechanics. The entanglement is implemented by a unitary transformation in the following way:¹ Define two unitary transformations, U_X and U_Y , that implement the X and Y measurements ‘at the quantum level’ on the tensor product of the Hilbert space of the channel

¹Note that there is no loss of generality in assuming that the channel particle is in a pure state. If the channel particle is entangled with Alice’s ancillas, the device implements the entanglement via the transformation $I \otimes \cdots$, where I is the identity operator in the Hilbert space of Alice’s ancillas.

particle, \mathcal{H}_C , and the Hilbert space of Bob's pointer ancilla, $\mathcal{H}_{B(P)}$:

$$\begin{aligned} |x_1\rangle_C |p_0\rangle_B &\xrightarrow{U_X} |x_1\rangle_C |p_1\rangle_B \\ |x_2\rangle_C |p_0\rangle_B &\xrightarrow{U_X} |x_2\rangle_C |p_2\rangle_B \end{aligned} \quad (1)$$

and

$$\begin{aligned} |y_1\rangle_C |p_0\rangle_B &\xrightarrow{U_Y} |y_1\rangle_C |p_1\rangle_B \\ |y_2\rangle_C |p_0\rangle_B &\xrightarrow{U_Y} |y_2\rangle_C |p_2\rangle_B \end{aligned} \quad (2)$$

so that

$$|\psi\rangle_C |p_0\rangle_B \xrightarrow{U_X} \langle x_1|\psi\rangle |x_1\rangle_C |p_1\rangle_B + \langle x_2|\psi\rangle |x_2\rangle_C |p_2\rangle_B \quad (3)$$

and

$$|\psi\rangle_C |p_0\rangle_B \xrightarrow{U_Y} \langle y_1|\psi\rangle |y_1\rangle_C |p_1\rangle_B + \langle y_2|\psi\rangle |y_2\rangle_C |p_2\rangle_B \quad (4)$$

The random choice is defined similarly by a unitary transformation V on the tensor product of the Hilbert space of Bob's die ancilla, $\mathcal{H}_{B(D)}$, and the Hilbert space $\mathcal{H}_C \otimes \mathcal{H}_{B(P)}$. Suppose $|d_X\rangle$ and $|d_Y\rangle$ are two orthogonal states in $\mathcal{H}_{B(D)}$ and that $|d_0\rangle = \frac{1}{\sqrt{2}}|d_X\rangle + \frac{1}{\sqrt{2}}|d_Y\rangle$. Then (suppressing the obvious subscripts) V is defined by:

$$\begin{aligned} |d_X\rangle \otimes |\psi\rangle |p_0\rangle &\xrightarrow{V} |d_X\rangle \otimes U_X |\psi\rangle |p_0\rangle \\ |d_Y\rangle \otimes |\psi\rangle |p_0\rangle &\xrightarrow{V} |d_Y\rangle \otimes U_Y |\psi\rangle |p_0\rangle \end{aligned} \quad (5)$$

so that

$$\begin{aligned} |d_0\rangle \otimes |\psi\rangle |p_0\rangle &\xrightarrow{V} \\ &\frac{1}{\sqrt{2}} |d_X\rangle \otimes U_X |\psi\rangle |p_0\rangle + \frac{1}{\sqrt{2}} |d_Y\rangle \otimes U_Y |\psi\rangle |p_0\rangle \end{aligned} \quad (6)$$

where the tensor product symbol has been introduced selectively to indicate that U_x and U_y are defined on $\mathcal{H}_C \otimes \mathcal{H}_{B(P)}$.

If Bob were to actually choose the observable X or Y randomly, and actually perform the measurement and obtain a particular eigenvalue, Alice's density operator for the channel particle would be:

$$\begin{aligned} &\frac{1}{2}(|\langle x_1|\psi\rangle|^2 |x_1\rangle\langle x_1| + |\langle x_2|\psi\rangle|^2 |x_2\rangle\langle x_2|) \\ &\frac{1}{2}(|\langle y_1|\psi\rangle|^2 |y_1\rangle\langle y_1| + |\langle y_2|\psi\rangle|^2 |y_2\rangle\langle y_2|) \end{aligned} \quad (7)$$

assuming that Alice does not know what observable Bob chose to measure, nor what outcome he obtained. But this is precisely the same density operator generated by tracing over Bob's ancilla particles for the state produced in (6). In other words, the density operator for the channel particle is the same for Alice, whether Bob randomly chooses which observable to measure and actually performs the measurement, or whether he implements an EPR cheating strategy with his two ancillas that produces the transition (6) on the enlarged Hilbert space.

If Bob is required to eventually report what measurement he performed and what outcome he obtained, he can at that stage measure the die ancilla for the eigenstate $|d_X\rangle$ or $|d_Y\rangle$, and then measure the pointer ancilla for the eigenstate $|p_1\rangle$ or $|p_2\rangle$. In effect, if we consider the ensemble of possible outcomes for the two measurements, Bob will have converted the 'improper' mixture generated by tracing over his ancillas to a 'proper' mixture. But the difference between a proper and improper mixture is undetectable by Alice since she has no access to Bob's ancillas, and it is only by measuring the composite system consisting of the channel particle together with Bob's ancillas that Alice could ascertain that the channel particle is entangled with the ancillas.

In fact, if it were possible to distinguish between a proper and improper mixture, it would be possible to signal superluminally: Alice could know instantaneously whether or not Bob performed a measurement on his ancillas by monitoring the channel particles in her possession. Note that it makes no difference whether Bob or Alice measures first, since the measurements are of observables in different Hilbert spaces, which therefore commute.

Clearly, a similar argument applies if Bob is required to choose between alternative unitary operations at some stage of a bit commitment protocol. Perhaps less obviously, an EPR cheating strategy is also possible if Bob is required to perform a measurement or choose between alternative operations on channel particle $i + 1$, conditional on the outcome of a prior measurement on channel particle i , or conditional on a prior choice of some operation from among a set of alternative operations. Of course, if Bob is in possession of all the channel particles at the same time, he can perform an entanglement with ancillas on the entire sequence, considered as a single composite system. But even if Bob only has access to one channel particle at a time (which he is required to return to Alice after performing a measurement or other operation before she sends him the next channel particle), he can always entangle channel particle $i + 1$ with the ancillas he used to entangle channel

particle i .

For example, suppose Bob is presented with two channel particles in sequence. He is supposed to decide randomly whether to measure X or Y on the first particle, perform the measurement, and return the particle to Alice. After Alice receives the first particle, she sends Bob the second particle. If Bob measured X on the first particle and obtained the outcome x_1 , he is supposed to measure X on the second particle; if he obtained the outcome x_2 , he is supposed to measure Y on the second particle. If he measured Y on the first particle and obtained the outcome y_1 , he is supposed to apply the unitary transformation U_1 to the second particle; if he obtained the outcome y_2 , he is supposed to apply the unitary transformation U_2 . After performing the required operation, he is supposed to return the second particle to Alice.

It would seem at first sight that Bob has to actually perform a measurement on the first channel particle and obtain a particular outcome before he can apply the protocol to the second particle, given that he only has access to one channel particle at a time, so an EPR cheating strategy is excluded. But this is not so. Bob's strategy is the following: He applies the EPR strategy discussed above for two alternative measurements to the first channel particle. For the second channel particle, he applies the following unitary transformation on the tensor product of the Hilbert spaces of his ancillas and the channel particle, where the state of the second channel particle is denoted by $|\phi\rangle$, and the state of the pointer ancilla for the second channel particle is denoted by $|q_0\rangle$ (a second die particle is not required):

$$\begin{aligned}
|d_X\rangle|p_1\rangle|\phi\rangle|q_0\rangle &\xrightarrow{U_C} |d_X\rangle|p_1\rangle \otimes U_X|\phi\rangle|q_0\rangle \\
|d_X\rangle|p_2\rangle|\phi\rangle|q_0\rangle &\xrightarrow{U_C} |d_X\rangle|p_2\rangle \otimes U_Y|\phi\rangle|q_0\rangle \\
|d_Y\rangle|p_1\rangle|\phi\rangle|q_0\rangle &\xrightarrow{U_C} |d_Y\rangle|p_1\rangle \otimes U_1|\phi\rangle \otimes |q_0\rangle \\
|d_Y\rangle|p_2\rangle|\phi\rangle|q_0\rangle &\xrightarrow{U_C} |d_Y\rangle|p_2\rangle \otimes U_2|\phi\rangle \otimes |q_0\rangle
\end{aligned} \tag{8}$$

Since an EPR cheating strategy can always be applied without detection, the proof of the bit commitment theorem assumes that at the end of the commitment stage the composite system consisting of Alice's ancillas, the n channel particles, and Bob's ancillas will be represented by some composite entangled state $|0\rangle$ or $|1\rangle$, depending on Alice's commitment, on a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, where \mathcal{H}_A is the Hilbert space of the particles in Alice's possession at that stage (Alice's ancillas and the channel particles retained by

Alice, if any), and \mathcal{H}_B is the Hilbert space of the particles in Bob's possession at that stage (Bob's ancillas and the channel particles retained by Bob, if any).

Now, the density operators $W_B(0)$ and $W_B(1)$ characterizing the information available to Bob for the two alternative commitments are obtained by tracing the states $|0\rangle$ and $|1\rangle$ over \mathcal{H}_A . If these density operators are the same, then Bob will be unable to distinguish the 0-commitment from the 1-commitment without further information from Alice. In this case, the protocol is said to be 'concealing.' What the proof establishes, by an application of the biorthogonal decomposition theorem, is that if $W_B(0) = W_B(1)$ then there exists a unitary transformation in \mathcal{H}_A that will transform $|0\rangle$ to $|1\rangle$. That is, if the protocol is 'concealing' then it cannot be 'binding' on Alice: she can always make the 0-commitment and follow the protocol (with appropriate applications of an EPR strategy) to establish the state $|0\rangle$. At the final stage when she is required to reveal her commitment, she can change her commitment if she chooses, depending on circumstances, by applying a suitable unitary transformation in her own Hilbert space to transform $|0\rangle$ to $|1\rangle$ without Bob being able to detect this move. So either Bob can cheat by obtaining some information about Alice's choice before she reveals her commitment, or Alice can cheat.

The essentials of the proof can be sketched as follows: In the biorthogonal (Schmidt) decomposition, the states $|0\rangle$ and $|1\rangle$ can be expressed as:

$$\begin{aligned} |0\rangle &= \sum_i \sqrt{c_i} |a_i\rangle |b_i\rangle \\ |1\rangle &= \sum_j \sqrt{c'_j} |a'_j\rangle |b'_j\rangle \end{aligned} \quad (9)$$

where $\{|a_i\rangle\}, \{|a'_j\rangle\}$ are two orthonormal sets of states in \mathcal{H}_A , and $\{|b_i\rangle\}, \{|b'_j\rangle\}$ are two orthonormal sets in \mathcal{H}_B .

The density operators $W_B(0)$ and $W_B(1)$ are defined by:

$$\begin{aligned} W_B(0) &= Tr_A |0\rangle\langle 0| = \sum_i c_i |b_i\rangle\langle b_i| \\ W_B(1) &= Tr_A |1\rangle\langle 1| = \sum_j c'_j |b'_j\rangle\langle b'_j| \end{aligned} \quad (10)$$

Bob can't cheat if and only if $W_B(0) = W_B(1)$. Now, by the spectral theorem, the decompositions:

$$W_B(0) = \sum_i c_i |b_i\rangle\langle b_i|$$

$$W_B(1) = \sum_j c'_j |b'_j\rangle\langle b'_j|$$

are unique. For the nondegenerate case, where the c_i are all distinct and the c'_j are all distinct, the condition $W_B(0) = W_B(1)$ implies that for all k :

$$\begin{aligned} c_k &= c'_k \\ |b_k\rangle &= |b'_k\rangle \end{aligned} \tag{11}$$

and so

$$\begin{aligned} |0\rangle &= \sum_k \sqrt{c_k} |a_k\rangle |b_k\rangle \\ |1\rangle &= \sum_k \sqrt{c_k} |a'_k\rangle |b_k\rangle \end{aligned} \tag{12}$$

It follows that there exists a unitary transformation $U \in \mathcal{H}_A$ such that

$$\{|a_k\rangle\} \xrightarrow{U} \{|a'_k\rangle\} \tag{13}$$

and hence

$$|0\rangle \xrightarrow{U} |1\rangle \tag{14}$$

The degenerate case can be handled in a similar way. Suppose that $c_1 = c_2 = c'_1 = c'_2 = c$. Then $|b_1\rangle, |b_2\rangle$ and $|b'_1\rangle, |b'_2\rangle$ span the same subspace \mathcal{H} in \mathcal{H}_B , and hence (assuming the coefficients are distinct for $k > 2$:

$$\begin{aligned} |0\rangle &= \sqrt{c}(|a_1\rangle|b_1\rangle + |a_2\rangle|b_2\rangle) + \sum_{k>2} \sqrt{c_k} |a_k\rangle |b_k\rangle \\ |1\rangle &= \sqrt{c}(|a'_1\rangle|b'_1\rangle + |a'_2\rangle|b'_2\rangle) + \sum_{k>2} \sqrt{c_k} |a'_k\rangle |b_k\rangle \\ &= \sqrt{c}(|a''_1\rangle|b_1\rangle + |a''_2\rangle|b_2\rangle) + \sum_{k>2} \sqrt{c_k} |a'_k\rangle |b_k\rangle \end{aligned} \tag{15}$$

where $|a''_1\rangle, |a''_2\rangle$ are orthonormal states spanning \mathcal{H} . Since $\{|a''_1\rangle, |a''_2\rangle, |a_3\rangle, \dots\}$ is an orthonormal set in \mathcal{H}_A , there exists a unitary transformation in \mathcal{H}_A that transforms $\{|a_k\rangle\}$ to $\{|a''_1\rangle, |a''_2\rangle, |a'_3\rangle, \dots\}$, and hence $|0\rangle$ to $|1\rangle$

The extension of the theorem to the nonideal case, where $W_B(0) \approx W_B(1)$, so that there is a small probability of Bob distinguishing the alternative commitments, shows that Alice has a correspondingly large probability

of cheating successfully: there exists a U that will transform $W_B(0)$ sufficiently close to $W_B(1)$ so that Bob has a very small probability of making the distinction.

The heart of the mathematical proof is the biorthogonal decomposition theorem. But the essential conceptual insight is the possibility of enlarging the Hilbert space and implementing an EPR strategy without detection. This raises the following question, considered in the next section: Suppose Bob cannot cheat because $W_B(0) = W_B(1)$, so by the theorem there exists a unitary transformation U in \mathcal{H}_A that will transform $|0\rangle$ to $|1\rangle$. Could there be a protocol in which Alice also cannot cheat because, although there exists a suitable unitary transformation U , she cannot know what unitary transformation to apply? In the next section we shall see that this is indeed the case, but only if U depends on Bob's operations, which are unknown to Alice. But then Bob would have to actually make a determinate choice or obtain a determinate outcome in a measurement, and he could always avoid doing so without detection by applying an EPR strategy. The remaining question would seem to be whether he might choose to avoid an EPR strategy in a certain situation because it would be disadvantageous to him. How do we know that following an EPR strategy is never disadvantageous?

3 A Possible Loophole?

The question at issue in this section is whether applying an EPR cheating strategy can ever be disadvantageous to the cheater. Note that the standard approach in cryptology is to consider the possibility of cheating against an honest opponent. Here we are considering the question of whether a quantum bit commitment protocol exists with the feature that one of the parties would forego a certain cheating strategy, because the opposing party would be able to cheat by taking advantage of such a move. So, strictly speaking, this would not be considered a loophole in the quantum bit commitment theorem, even if we could identify such a protocol. Nevertheless, this 'game-theoretic' extension of the usual notion is certainly relevant to the issue of security.

To focus the question, it will be worthwhile to consider a particular protocol based on the Aharonov-Bergmann-Lebowitz notion of pre- and post-selected quantum states [1]. If (i) Alice prepares a system in a certain state $|\text{pre}\rangle$ at time t_1 , (ii) Bob measures some observable Q on the system at time t_2 , and (iii) Alice measures an observable of which $|\text{post}\rangle$ is an eigenstate at

time t_3 , and post-selects for $|\text{post}\rangle$, then Alice can assign probabilities to the outcomes of Bob's Q -measurement at t_2 , conditional on the states $|\text{pre}\rangle$ and $|\text{post}\rangle$ at times t_1 and t_3 , respectively, as follows:

$$\text{prob}(q_k) = \frac{|\langle \text{pre} | P_k | \text{post} \rangle|^2}{\sum_i |\langle \text{pre} | P_i | \text{post} \rangle|^2} \quad (16)$$

where P_i is the projection operator onto the i 'th eigenspace of Q . Notice that the *ABL*-rule is time-symmetric, in the sense that the states $|\text{pre}\rangle$ and $|\text{post}\rangle$ can be interchanged, so these states are sometimes referred to as time-symmetric states.

If Q is unknown to Alice, she can use this 'ABL-rule' to assign probabilities to the outcomes of various hypothetical Q -measurements. The interesting peculiarity of the ABL-rule, by contrast with the usual Born rule for pre-selected states, is that it is possible—for an appropriate choice of observables Q, Q', \dots , and states $|\text{pre}\rangle$ and $|\text{post}\rangle$ —to assign unit probability to the outcomes of a set of mutually *noncommuting* observables. That is, Alice can be in a position to assert a conjunction of conditional statements of the form: 'If Bob measured Q , then the outcome must have been q_i , with certainty, and if Bob measured Q' , then the outcome must have been q'_j , with certainty, \dots ,' where Q, Q', \dots are mutually noncommuting observables.

A case of this sort has been discussed by Vaidman, Aharonov, and Albert [30], where the outcome of a measurement of any of the three spin components $\sigma_x, \sigma_y, \sigma_z$ of a spin- $\frac{1}{2}$ particle can be inferred from an appropriate pre- and post-selection. Alice prepares a pair of particles, A and C , in the Bell state:

$$|\text{pre}\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle_A |\uparrow_z\rangle_C + |\downarrow_z\rangle_A |\downarrow_z\rangle_C) \quad (17)$$

where $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$ denote the σ_z -eigenstates. Alice sends the channel particle C to Bob and keeps the ancilla A . Bob measures either σ_x , or σ_y , or σ_z on the channel particle and returns the channel particle to Alice. Alice then measures an observable R on the pair of particles, where R has the eigenstates:

$$|r_1\rangle = \frac{1}{\sqrt{2}}|\uparrow_z\rangle|\uparrow_z\rangle + \frac{1}{2}(|\uparrow_z\rangle|\downarrow_z\rangle e^{i\pi/4} + |\downarrow_z\rangle|\uparrow_z\rangle e^{-i\pi/4}) \quad (18)$$

$$|r_2\rangle = \frac{1}{\sqrt{2}}|\uparrow_z\rangle|\uparrow_z\rangle - \frac{1}{2}(|\uparrow_z\rangle|\downarrow_z\rangle e^{i\pi/4} + |\downarrow_z\rangle|\uparrow_z\rangle e^{-i\pi/4}) \quad (19)$$

$$|r_3\rangle = \frac{1}{\sqrt{2}}|\downarrow_z\rangle|\downarrow_z\rangle + \frac{1}{2}(|\uparrow_z\rangle|\downarrow_z\rangle e^{-i\pi/4} + |\downarrow_z\rangle|\uparrow_z\rangle e^{i\pi/4}) \quad (20)$$

$$|r_4\rangle = \frac{1}{\sqrt{2}}|\downarrow_z\rangle|\downarrow_z\rangle - \frac{1}{2}(|\uparrow_z\rangle|\downarrow_z\rangle e^{-i\pi/4} + |\downarrow_z\rangle|\uparrow_z\rangle e^{i\pi/4}) \quad (21)$$

Note that:

$$|\text{pre}\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle|\uparrow_z\rangle + |\downarrow_z\rangle|\downarrow_z\rangle) \quad (22)$$

$$= \frac{1}{\sqrt{2}}(|\uparrow_x\rangle|\uparrow_x\rangle + |\downarrow_x\rangle|\downarrow_x\rangle) \quad (23)$$

$$= \frac{1}{\sqrt{2}}(|\uparrow_y\rangle|\downarrow_y\rangle + |\downarrow_y\rangle|\uparrow_y\rangle) \quad (24)$$

$$= \frac{1}{2}(|r_1\rangle + |r_2\rangle + |r_3\rangle + |r_4\rangle) \quad (25)$$

Alice can now assign values to the outcomes of Bob's spin measurements via the ABL-rule, whether Bob measured σ_x , σ_y , or σ_z , based on the post-selections $|r_1\rangle$, $|r_2\rangle$, $|r_3\rangle$, or $|r_4\rangle$, according to Table 1.

| | σ_x | σ_y | σ_z |
|-------|--------------|--------------|--------------|
| r_1 | \uparrow | \uparrow | \uparrow |
| r_2 | \downarrow | \downarrow | \uparrow |
| r_3 | \uparrow | \downarrow | \downarrow |
| r_4 | \downarrow | \uparrow | \downarrow |

Table 1: σ_x , σ_y , σ_z measurement outcomes correlated with eigenvalues of R

Consider, now, the following protocol for bit commitment based on the Vaidman-Aharonov-Albert case. Alice prepares n copies of the Bell state $|\text{pre}\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle_A|\uparrow_z\rangle_C + |\downarrow_z\rangle_A|\downarrow_z\rangle_C)$. She keeps the ancillas and sends the channel particles to Bob in sequence. Bob measures either σ_x , σ_y , or σ_z chosen randomly on a channel particle, records the outcome, and returns the particle to Alice before she sends him the next channel particle in the sequence. Alice measures the observable R on each channel particle she receives back from Bob.

The commitment is made as follows: After the sequence of measurements, Bob announces the indices in the sequence for which he obtained a ' \uparrow ' outcome for his measurements (without announcing whether he measured σ_x ,

σ_y , or σ_z). The remaining elements in the sequence are discarded. Alice can now divide the \uparrow -sequence into two subsequences of approximately equal length (for large n): the subsequence S_1 for which she obtained the outcome r_1 for R , and the complementary subsequence S_{234} for which she obtained the outcome r_2, r_3 , or r_4 . If Alice commits to 0, she announces the indices of the subsequence S_{234} and proves her commitment at the final stage, when she reveals her commitment, by her ability to announce (from Table 1), for each element in the subsequence, the observable that Bob measured, either σ_x, σ_y , or σ_z . If she commits to 1, she announces the indices of the subsequence S_1 and proves her commitment by her ability to announce, for each element in the *complementary* subsequence S_{234} , the observable that Bob measured.

At first sight, it might appear that this protocol is not of the sort covered by the bit commitment theorem. To see that it is, suppose that instead of following the protocol and actually choosing one of σ_x, σ_y , or σ_z , performing the measurement, and obtaining a determinate outcome, Bob implements an EPR cheating strategy with a quantum die ancilla with three orthogonal states $|d_x\rangle, |d_y\rangle, |d_z\rangle$ corresponding to the choice of spin observable $\sigma_x, \sigma_y, \sigma_z$. Then the state of the composite system consisting of Alice's ancilla, the channel particle, and Bob's die and pointer ancillas is:

$$\begin{aligned} |\Psi\rangle = & \frac{1}{\sqrt{3}}|d_x\rangle_B \left(\frac{1}{\sqrt{2}}|\uparrow_x\rangle_A |\uparrow_x\rangle_C |p_\uparrow\rangle_B + \frac{1}{\sqrt{2}}|\downarrow_x\rangle_A |\downarrow_x\rangle_C |p_\downarrow\rangle_B \right) \\ & + \frac{1}{\sqrt{3}}|d_y\rangle_B \left(\frac{1}{\sqrt{2}}|\uparrow_y\rangle_A |\downarrow_y\rangle_C |p_\downarrow\rangle_B + \frac{1}{\sqrt{2}}|\downarrow_y\rangle_A |\uparrow_y\rangle_C |p_\uparrow\rangle_B \right) \\ & + \frac{1}{\sqrt{3}}|d_z\rangle_B \left(\frac{1}{\sqrt{2}}|\uparrow_z\rangle_A |\uparrow_z\rangle_C |p_\uparrow\rangle_B + \frac{1}{\sqrt{2}}|\downarrow_z\rangle_A |\downarrow_z\rangle_C |p_\downarrow\rangle_B \right) \end{aligned} \quad (26)$$

To announce ' \uparrow ,' Bob measures the pointer ancilla for p_\uparrow or p_\downarrow , which projects $|\Psi\rangle$ onto:

$$|\uparrow\rangle = |p_\uparrow\rangle_B \frac{1}{\sqrt{3}}(|d_x\rangle_B |\uparrow_x\rangle_A |\uparrow_x\rangle_C + |d_y\rangle_B |\downarrow_y\rangle_A |\uparrow_y\rangle_C + |d_z\rangle_B |\uparrow_z\rangle_A |\uparrow_z\rangle_C) \quad (27)$$

or

$$|\downarrow\rangle = |p_\downarrow\rangle_B \frac{1}{\sqrt{3}}(|d_x\rangle_B |\downarrow_x\rangle_A |\downarrow_x\rangle_C + |d_y\rangle_B |\uparrow_y\rangle_A |\downarrow_y\rangle_C + |d_z\rangle_B |\downarrow_z\rangle_A |\downarrow_z\rangle_C) \quad (28)$$

with probability $\frac{1}{2}$. Note that this enables Bob to announce the ' \uparrow ' outcomes without actually measuring σ_x, σ_y , or σ_z ! In effect, he has a quantum

computer that computes ‘ \uparrow ’ or ‘ \downarrow ’ for the quantum disjunction ‘ σ_x or σ_y or σ_z .’

The state $|\uparrow\rangle$ can be expressed in terms of R -eigenstates:

$$\begin{aligned} |\uparrow\rangle &= \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{2}}|r_1\rangle_A + \frac{1}{\sqrt{2}}|r_3\rangle_A\right)|d_x\rangle_B|p_\uparrow\rangle_B \\ &\quad + \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{2}}|r_1\rangle_A + \frac{1}{\sqrt{2}}|r_4\rangle_A\right)|d_y\rangle_B|p_\uparrow\rangle_B \\ &\quad + \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{2}}|r_1\rangle_A + \frac{1}{\sqrt{2}}|r_2\rangle_A\right)|d_z\rangle_B|p_\uparrow\rangle_B \end{aligned} \quad (29)$$

and rewritten as:

$$\begin{aligned} |\uparrow\rangle &= \frac{1}{\sqrt{2}}|r_1\rangle_A\left(\frac{1}{\sqrt{3}}|d_x\rangle_B + \frac{1}{\sqrt{3}}|d_y\rangle_B + \frac{1}{\sqrt{3}}|d_z\rangle_B\right)|p_\uparrow\rangle_B \\ &\quad + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{3}}|r_3\rangle_A|d_x\rangle_B + \frac{1}{\sqrt{3}}|r_4\rangle_A|d_y\rangle_B + \frac{1}{\sqrt{3}}|r_2\rangle_A|d_z\rangle_B\right)|p_\uparrow\rangle_B \end{aligned} \quad (30)$$

Evidently, after Alice measures the observable R on the channel particles in the ‘ \uparrow ’ subsequence and announces either the subsequence S_{234} for which she obtained the eigenvalues r_2 , r_3 , or r_4 corresponding to the 0-commitment, or the subsequence S_1 for which she obtained the eigenvalue r_1 corresponding to the 1-commitment, Bob’s density operator for the channel particles (obtained by tracing over Alice’s ancillas and Bob’s ancillas) will be either:

$$W_B(0) = \frac{1}{3}(|d_x\rangle_{BB}\langle d_x| + |d_y\rangle_{BB}\langle d_y| + |d_z\rangle_{BB}\langle d_z|) \quad (31)$$

for the subsequence S_{234} , or:

$$W_B(1) = \frac{1}{3}(|d_x\rangle_B + |d_y\rangle_B + |d_z\rangle_B)({}_B\langle d_x| + {}_B\langle d_y| + {}_B\langle d_z|) \quad (32)$$

for the subsequence S_1 . (More precisely, these are the density operators for a single channel particle. The density operator for the sequence of channel particles is in each case a tensor product of the relevant operator over the elements of the sequence.) But these density operators are distinguishable: W_0 is the density operator of an equal weight mixture of pure states $|d_x\rangle_B$, $|d_y\rangle_B$, and $|d_z\rangle_B$, while W_1 is the density operator of the pure state $\frac{1}{\sqrt{3}}|d_x\rangle_B + \frac{1}{\sqrt{3}}|d_y\rangle_B + \frac{1}{\sqrt{3}}|d_z\rangle_B$. So Bob can cheat—the protocol is insecure.

Now, suppose we assume that Bob is forced to make a determinate choice of which spin component observable to measure for each channel particle, and actually perform the measurements and record the outcomes. Then it is clear that both subsequences S_1 and S_{234} will be characterized by the same equal weight mixture of pure states $|d_x\rangle_B$, $|d_y\rangle_B$, and $|d_z\rangle_B$. So Bob cannot cheat. But Alice cannot cheat either. Of course, by the bit commitment theorem, since Alice is in possession of all the channel particles at the final stage when she is required to reveal her commitment, there exists a unitary transformation in Alice's Hilbert space (which now includes the channel particles) that will transform the states of the ancilla-channel pairs to R -eigenstates that conform to Bob's measurement outcomes. But this unitary transformation depends on the outcomes of Bob's measurements, which are unknown to Alice. Essentially, Alice would have to transform the state $|r_1\rangle$ for each element in the declared subsequence or the complementary subsequence to the state $|r_2\rangle$, $|r_3\rangle$, or $|r_4\rangle$, corresponding to Bob's measurement outcome for that element, in order to successfully change her commitment without Bob being able to detect her cheating. There exists a unitary transformation that Alice can implement to achieve this result, but she cannot know what unitary transformation to employ. So the protocol is secure, subject to the assumption that Bob cannot apply an EPR cheating strategy.

The question raised at the beginning of this section can now be put more concretely. In the above protocol, if Bob is honest and does not apply an EPR strategy, then neither party can cheat. If he applies the strategy, then he gains the advantage. Can there be a bit commitment protocol that is similar to the above protocol, except that the application of an EPR strategy by Bob at a certain stage of the protocol would give Alice the advantage, rather than Bob, while conforming to the protocol would ensure that neither party could cheat? If there were such a protocol, then Bob would, in effect, be forced to conform to the protocol and avoid the EPR strategy, and unconditionally secure bit commitment would be possible.

In fact, the impossibility of such a protocol follows from the theorem itself. Suppose there were such a protocol. That is, suppose that if Bob applies an EPR strategy then $W_B(0) = W_B(1)$, so by the theorem there exists a unitary transformation U in Alice's Hilbert space that will transform $|0\rangle$ to $|1\rangle$. Alice must know this U because it is uniquely determined by Bob's deviation from the protocol according to an EPR strategy that keeps all disjunctions at the quantum level as linear superpositions. Suppose also that if, instead, Bob is honest and follows the protocol (so that there is a determinate choice for every

disjunction over possible operations or possible measurement outcomes), then $W_B(0) = W_B(1)$, but the unitary transformation in Alice's Hilbert space that allows her to transform $|0\rangle$ to $|1\rangle$ depends on Bob's choices or measurement outcomes, which are unknown to Alice.

Now the crucial point to note is that the information available in Alice's Hilbert space must be the same whether Bob follows the protocol and makes determinate choices and obtains determinate measurement outcomes before Alice applies the unitary transformation U that transforms $|0\rangle$ to $|1\rangle$, or whether he deviates from the protocol via an EPR strategy in which he implements corresponding entanglements with his ancillas to keep choices and measurement outcomes at the quantum level before Alice applies the transformation U , and only makes these choices and measurement outcomes determinate at the final stage of the protocol by measuring his ancillas. There can be no difference for Alice because Bob's measurements on his ancillas and any measurements or operations that Alice might perform take place in different Hilbert spaces, so the operations commute. If Alice's density operator (obtained by tracing over Bob's ancillas), which characterizes the statistics of measurements that Alice can perform in her part of the universe, were different depending on whether or not Bob actually carried out the required measurements, as opposed to keeping the alternatives at the quantum level by implementing corresponding entanglements with ancillas, then it would be possible to use this difference to signal superluminally. Actual measurements by Bob on his ancillas that selected alternatives in the entanglements as determinate would instantaneously alter the information available in Alice's part of the universe.

It follows that in the hypothetical bit commitment protocol we are considering, the unitary transformation U in Alice's Hilbert space that transforms $|0\rangle$ to $|1\rangle$ must be the same transformation in the honest scenario as in the cheating scenario. But we are assuming that the transformation in the honest scenario is unknown to Alice and depends on Bob's measurement outcomes, while the transformation in the cheating scenario is unique and known to Alice. So there can be no such protocol: the deviation from the protocol by an EPR strategy can never place Bob in a worse position than following the protocol honestly.

The argument can be put formally in terms of the theorem as follows: The cheating scenario produces one of two alternative pure states $|0\rangle_c$ or $|1\rangle_c$ in $\mathcal{H}_A \otimes \mathcal{H}_B$ ('c' for 'cheating strategy'). Since the reduced density operators

in \mathcal{H}_B :

$$\begin{aligned} W_B^{(c)}(0) &= Tr_A |0\rangle_{cc} \langle 0| \\ W_B^{(c)}(1) &= Tr_A |1\rangle_{cc} \langle 1| \end{aligned} \quad (33)$$

are required by assumption to be the same:

$$W_B^{(c)}(0) = W_B^{(c)}(1) \quad (34)$$

the states $|0\rangle_c$ and $|1\rangle_c$ can be expressed in biorthogonal decomposition as:

$$\begin{aligned} |0\rangle_c &= \sum_i \sqrt{c_i} |a_i\rangle |b_i\rangle \\ |1\rangle_c &= \sum_i \sqrt{c_i} |a'_i\rangle |b_i\rangle \end{aligned} \quad (35)$$

where the reduced density operators in \mathcal{H}_A :

$$\begin{aligned} W_A^{(c)}(0) &= Tr_B |0\rangle_{cc} \langle 0| = \sum_i |c_i| a_i \rangle \langle a_i| \\ W_A^{(c)}(1) &= Tr_B |1\rangle_{cc} \langle 1| = \sum_i |c_i| a'_i \rangle \langle a'_i| \end{aligned} \quad (36)$$

are different:

$$W_A^{(c)}(0) \neq W_A^{(c)}(1) \quad (37)$$

It follows that there exists a unitary operator $U_c \in \mathcal{H}_A$ defined by the spectral representations of $W_A^{(c)}(0)$ and $W_A^{(c)}(1)$:

$$\{|a_i\rangle\} \xrightarrow{U_c} \{|a'_i\rangle\} \quad (38)$$

such that:

$$|0\rangle_c \xrightarrow{U_c} |1\rangle_c \quad (39)$$

The honest scenario produces one of two alternative pure states $|0\rangle_h$ and $|1\rangle_h$ in $\mathcal{H}_A \otimes \mathcal{H}_B$ ('h' for 'honest scenario'), where the pair $\{|0\rangle_h, |1\rangle_h\}$ depends on Bob's choices and the outcomes of his measurements.

By assumption, as in the cheating scenario, the reduced density operators $W_B^{(h)}(0)$ and $W_B^{(h)}(1)$ in \mathcal{H}_B are the same:

$$W_B^{(h)}(0) = W_B^{(h)}(1) \quad (40)$$

which entails the existence of a unitary operator $U_h \in \mathcal{H}_A$ such that:

$$|0\rangle_h \xrightarrow{U_h} |1\rangle_h \quad (41)$$

where U_h depends on Bob's choices and measurement outcomes.

Now, the difference between the honest scenario and the cheating scenario is undetectable in \mathcal{H}_A , which means that the reduced density operators in \mathcal{H}_A are the same in the honest scenario as in the cheating scenario:

$$\begin{aligned} W_A^{(h)}(0) &= W_A^{(c)}(0) \\ W_A^{(h)}(1) &= W_A^{(c)}(1) \end{aligned} \quad (42)$$

Since U_h is defined by the spectral representations of $W_A^{(h)}(0)$ and $W_A^{(h)}(1)$, it follows that $U_h = U_c$. But we are assuming that U_h depends on Bob's choices and measurement outcomes, while U_c is uniquely defined by Bob's EPR strategy, in which there are no determinate choices or measurement outcomes. Conclusion: there can be no bit commitment protocol in which neither Alice nor Bob can cheat if Bob honestly follows the protocol, but Alice can cheat if Bob deviates from the protocol via an EPR strategy. If neither Bob nor Alice can cheat in the honest scenario, then Bob and not Alice must be able to cheat in the cheating scenario.

A similar argument rules out a protocol in which neither party can cheat if Bob is honest (as above), but if Bob follows an EPR strategy, then $W_B(0) \approx W_B(1)$, so Bob has some probability of cheating successfully, but Alice has a greater probability of cheating successfully than Bob. Again, the unitary transformation U_c that would allow Alice to cheat with a certain probability of success if Bob followed an EPR strategy would also have to allow Alice to cheat successfully if Bob were honest. But the supposition is that Alice cannot cheat if Bob is honest, because the unitary transformation U_h in that case depends on Bob's choices and measurement outcomes, which are unknown to Alice. It follows that there can be no such protocol.

So there is no loophole – not even in the extended sense: following an EPR cheating strategy can never be disadvantageous to the cheater. Unconditionally secure quantum bit commitment (in the sense of the theorem) really is impossible.

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